MA 401 Question and answers

Tutorial 1:

1. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= 0$; only one pivot ; only one pivot ; $= I$

A : Row reduced → see the defn of row-reduced echelon metric

2. $A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $Ax = y$

let $A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix} \xrightarrow{R_3' = R_3 - \frac{1}{3}R_1} \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 0 & -\frac{8}{3} & -\frac{2}{3} \end{bmatrix} \xrightarrow{R_2' = R_2 - \frac{2}{3}R_1} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 5/3 & -1/3 \\ 0 & -8/3 & -2/3 \end{bmatrix}$

$\xrightarrow{R_3' = R_3 + \frac{8}{5}R_2} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 5/3 & -1/3 \\ 0 & 0 & -6/5 \end{bmatrix}$

$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \xrightarrow{R_3' = R_3 - \frac{1}{3}R_1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 - \frac{y_1}{3} \end{bmatrix} \xrightarrow{R_2' = R_2 - \frac{2}{3}R_1} \begin{bmatrix} y_1 \\ y_2 - \frac{2}{3}y_1 \\ y_3 - \frac{y_1}{3} \end{bmatrix} \xrightarrow{R_3' = R_3 + \frac{8}{5}R_2} \begin{bmatrix} y_1 \\ y_2 - \frac{2}{3}y_1 \\ y_3 - \frac{y_1}{3} + \frac{8}{5}\left(y_2 - \frac{2}{3}y_1\right) \end{bmatrix}$

now for $y_3 - \frac{y_1}{3} + \frac{8}{5}\left(y_2 - \frac{2}{3}y_1\right) \neq 0$ it has a unique soln.

$$y_3 - \frac{y_1}{3} + \frac{8}{5}y_2 - \frac{16}{15}y_1 \neq 0$$

$$y_3 + \frac{8}{5}y_2 - \frac{21}{15}y_1 \neq 0$$

$$y_3 + \frac{8}{5}y_2 - \frac{7}{5}y_1 \neq 0$$

$$5y_3 + 8y_2 - 7y_1 \neq 0$$

3. A $(n \times n)$ matrix

(P) $\exists\ B_{n \times n}$ s.t

$BA = I$

as $BA = I$

if $AB = I$ then

A is invertible.

as $BA = I$

$BAB = B$

$ABAB = AB$

$(AB)^2 = AB$

as $AB = (AB)^2$
$AB = (AB)^n \ \forall n \in \mathbb{N}$
$(AB)^2 - AB = 0$

$(AB)(AB - I) = 0$
$AB = 0$
or $AB = I$

but $AB \neq 0$ as if $AB = 0$
$AB \cdot A = 0$
$A = 0$
but as $BA = I$
$A \neq 0$ $\ast$

$\therefore AB = I = BA$
$\therefore A$ is invertible

(ii) $AC = I$ similar to (i).

4. $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}_{3\times 2}$ $B = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}_{2\times 2}$ To find C and if unique. $CA = B$

C has to be $_{2\times 3}$

let $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}$$

$c_{11} + 2c_{12} + c_{13} = 3$

$-c_{11} + 2c_{12} = 1$

$2c_{12} - 1 = c_{11}$

$2c_{12} - 1 + 2c_{12} + c_{13} = 3$

$4c_{12} + c_{13} = 4$

$c_{13} = 4 - 4c_{12}$
$c_{11} = 2c_{12} - 1$
$c_{12} = c_{12}$

$c_{21} + 2c_{22} + c_{23} = -4$
$-c_{21} + 2c_{22} = 4$
$2c_{22} - 4 = c_{21}$
$2c_{22} - 4 + 2c_{22} + c_{23} = -4$
$4c_{22} + c_{23} = 0$
$c_{23} = -4c_{22}$
$c_{22} = c_{22}$
$c_{21} = 2c_{22} - 4$

$$\begin{bmatrix} 2c_{12} - 1 & c_{12} & 4 - 4c_{12} \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2c_{22} - 4 & c_{22} & -4c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 4 \\ -4 & 0 & 0 \end{bmatrix} + c_{12} \begin{bmatrix} 2 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} + c_{22} \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & -4 \end{bmatrix}$$

5. $A_{m \times n}$

$B_{n \times m}$ $n < m$

AB is non-invertible.

AB is $m \times m$

AB $m \times m$ is non-invertible

as $A \times B = A_{1*} \cdot B_{*1} + A_{2*} B_{*2} + \cdots$

as A, B can have max n pivots

$$A \cdot B = \begin{bmatrix} A_{1*} \\ \vdots \\ A_{m*} \end{bmatrix} \cdot \begin{bmatrix} B_{*1} \cdots B_{*m} \end{bmatrix}$$

row reduced form of A with n pivots is:

$$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

for B:

$$\begin{bmatrix} 1 & & 0 \\ 0 & \ddots & \end{bmatrix}$$

$$r(A) \cdot r(B) = \begin{bmatrix} \underbrace{1 \ddots 1}_{n \times n} & \\ & \end{bmatrix}$$

so row reduced form of A.B can have max n pivots.

as $n < m$, AB is invertible

6. $B \in M_{n \times n}(\mathbb{C})$

(i) To prove: $B' = \{ A \in M_{n \times n}(\mathbb{C}) \mid AB = BA \}$ is a subspace.

proof: for $A, C \in B'$ $AB = BA$ and $CB = BC$
$\alpha \in \mathbb{C}$

if $\alpha A + C \in B'$ (proved)

now
$$\begin{aligned} (\alpha A + C)B &= \alpha AB + CB \\ &= \alpha BA + BC \\ &= B\alpha A + BC \\ &= B(\alpha A + C) \end{aligned}$$

$\therefore \alpha A + C \in B'$

$\therefore$ subspace

(ii) $B = \begin{bmatrix} 1 & & 0 \\ & 2 & \\ 0 & & \ddots n \end{bmatrix}_{n \times n}$ to find: $B' = \{ A \in M_{n \times n}(\mathbb{C}) \mid AB = BA \}$

for $B = \begin{bmatrix} 1 & & & 0 \\ & 2 & & \\ & & 3 & \\ 0 & & & \ddots n \end{bmatrix}_{n \times n}$

let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}_{n\times n}$ $\forall a_{ij} \in \mathbb{C}$, $A \in M_{n\times n}(\mathbb{C})$

now $AB = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ \vdots & & \vdots \\ \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & 2 & & \\ & & 3 & \\ 0 & & & n \end{bmatrix} = \begin{bmatrix} a_{11} & 2a_{12} & 3a_{13} & \cdots & na_{1n} \\ a_{21} & 2a_{22} & \cdots & \cdots & na_{2n} \\ \vdots & \vdots & & & na_{nn} \end{bmatrix}$

$BA = \begin{bmatrix} 1 & & & 0 \\ & 2 & & \\ & & 3 & \\ 0 & & & n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots \\ \vdots & & \vdots \\ \vdots & & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 2a_{21} & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ na_{n1} & na_{n2} & \cdots & na_{nn} \end{bmatrix}$

for $AB = BA$

a_{11}, a_{nn} can be anything in $\mathbb{C}$

as $2a_{21} = a_{21}$

$\Rightarrow a_{21} = 0$ (similarly for others)

similarly a_{ii} can be any value.

so $B' = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix}$

where diagonal element can be anything in $\mathbb{C}$

so $\forall a_i \in \mathbb{C}$. B'

7. $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in M_{n\times 1}(\mathbb{C})$

$$LA(X) = \{A \in M_{n\times n}(\mathbb{C}) \mid AX = 0\}$$

(i) for $LA(X)$ to be a subspace of $M_{n\times n}(\mathbb{C})$:

let $\alpha \in \mathbb{C}$

$A_1, A_2 \in LA(X)$

then $A_1X = 0$ and $A_2X = 0$

$\Rightarrow \alpha A_1 X = 0$ and $A_2 X = 0$

$\Rightarrow (\alpha A_1 + A_2)X = 0$

$\Rightarrow \alpha A_1 + A_2 \in LA(X)$

$\therefore$ proved.

(ii) <u>To prove</u>: $BA \in LA(X)$ whenever $A \in LA(X)$ and $B \in M_{n\times n}(\mathbb{C})$ (left ideal)

<u>proof</u>: as $A \in LA(X)$

$AX = 0$

now if $B \in M_{n\times n}(\mathbb{C})$

then $B.(Ax) = B.0 = 0$

$\Rightarrow (BA)x = 0 \Rightarrow BA \in LA(X)$

(iii) To prove: $LA(X) = M_{n\times n}(\mathbb{C}) \Leftrightarrow X = 0$

proof: ($\Rightarrow$) given $LA(X) = M_{n\times n}(\mathbb{C})$ then

$I \in LA(X)$

so $IX = 0$

$\Rightarrow X = 0$

($\Leftarrow$) $X = 0$, $\forall A \in M_{n\times n}(\mathbb{C})$

$A.0 = 0$

i.e $AX = 0$

so $M_{n\times n}(\mathbb{C}) = LA(X)$

8. $A, B \in M_{n\times n}(\mathbb{C})$ s.t $AB = 0$

(i) $BA = 0$

false: counter-example:

$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $AB = 0$

$BA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = BA \neq 0$

(ii) false (above)

(iii) Proof:

as B is invertible, $\exists C$ s.t $BC = I = CB$

$C = B^{-1}$

now, for $AB = 0$

$ABC = 0.C = 0$

$\Rightarrow A(BC) = 0$

$\Rightarrow A.I = 0$

$\Rightarrow A = 0$

(iv) for $x \neq 0$ but $BAx = 0$

BA have to be non invertible ($A, B \in M_{n\times n}(\mathbb{C})$)

as $|AB| = |BA| = 0$

BA is non-invertible

$\therefore$ Possible.

as $AB = 0$

$AB_{*1} = 0$

so for $x = B_{*1}$

so for $B_{*1} \neq 0$

for $x \neq 0$

works.

for $B_{*1} = 0$

$B_{*i} \neq 0$

if all $B_{*i} = 0$

$AB = 0$

then $A\underbrace{Bx}_{=0} = 0$

$\forall x \in \mathbb{R}^n$

Possible

$\therefore x \neq 0$

$ABx = 0$

Tutorial - 2 :

1. $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -6 & -4 & -2 \end{bmatrix}$

lin dep in $\mathbb{R}^4$

2. $V \in M_{2\times2}(\mathbb{C}) = V$

(i) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

can form $v = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} \in V$

Note: ① spans V
② lin ind $\Rightarrow$ Basis
③ $|\text{Basis}| = \dim$

$\therefore \dim = 4$

(ii) $W = \{A \in M_{2\times2}(\mathbb{C}) \mid A_{11} + A_{22} = 0\}$

$W = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \middle| \; a_{11} + a_{22} = 0 \right\}$

$= \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{bmatrix} \in M_{2\times2}(\mathbb{C}) \right\}$

$\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ can form W.

$\therefore \dim W = 3$

also if $\alpha \omega_1 + \beta \omega_2 \in W$ then ω is a subspace.

here $\alpha \omega_1 + \beta \omega_2$

$\alpha a_{11} + \beta b_{11} = w_{11}$
$-\alpha a_{11} - \beta b_{11} = w_{22}$

as $w_{11} + w_{22} = 0$

$\alpha \omega_1 + \beta \omega_2 \in W$

3. $X = \text{span}\{a, x, x^2, \ldots\ldots x^n\}$

$Y = \{ f(x) \in X \mid f(x) = 0 \text{ at } x = t_1, t_2, \ldots t_j \text{ for } j < n \}$

(i) $f(t_i) = 0$ also let
$f(x) \in Y$
and $g(x) \in Y$
then $\alpha f(x) + \beta g(x) \in Y$
as for $x = t_1, t_2, \ldots t_j$
both are zero
$\Rightarrow \alpha(0) + \beta(0) = 0$

X = set of poly deg $< n$

$Y = \{ \phi \in X \mid \phi(t_i) = 0, i \in \{1, 2, \ldots j\} \}$

$\phi(x) \in Y$

(ii) $f(x) = \alpha(x-t_1)(\cdot\cdot\cdot\cdot\cdot)$

$\downarrow$ $\underbrace{\qquad\qquad}$ remains same

$\alpha \in \mathbb{R}$ s.t

$\alpha \neq 0$

$\therefore \deg Y = 2$

as $\text{span } Y = \{\alpha, (x-t_1)(x-t_2)\cdots(x-t_j)\}$

(iii) $\dim(X/Y) = \{x Y \mid x \in X\}$

$\hookrightarrow$ polynomials with zero $= t_1, t_2, \ldots t_j$

$$\text{span}(X/Y) = \{\alpha, x, x^2, \ldots x^{n-j}, (x-t_1)(x-t_2)\cdots(x-t_j)\}$$

$$\deg(X/Y) = n-j+1+1 = n-j+2$$

then

$$\phi(x) = (x-t_1)\cdots(x-t_j)Q(x)$$

$$\deg(Q(x)) < n-j$$

$$B_Y = \left\{ \prod_{i=1}^{j}(x-x_i), \prod_{i=1}^{j}(x-x_i)x, \ldots, \prod_{i=1}^{j}(x-x_i)x^{n-j-1} \right\}$$

<u>Claim</u>: B_Y forms basis of Y

<u>B_Y spans Y</u>:

$$y \ni \phi(x) = \prod_{i=1}^{j}(x-x_i)\,Q(x)$$

where,

$$Q(x) = \sum_{k=0}^{n-j-1} b_k x^k$$

$$\phi(x) = \prod_{i=1}^{j}(x-x_i)\left(\sum_{k=0}^{n-j-1} b_k x^k\right)$$

$$= \sum_{k=0}^{n-j-1}\left\{\prod_{i=1}^{j}(x-x_i)\right\}x^k$$

4. V = vector space

A = lin ind subset of V

A = basis of V $\Leftrightarrow$ A is a maximal lin ind subset of V

① span $A = V$ ① any addition of vetor $\to$ lin dep

② A is lin ind

<u>proof</u>:

($\Rightarrow$) A = basis of V

span $A = V$

A is lin ind

let $\dim A = n$ so

$A = \{v_1, v_2, \ldots v_n\}$

s.t if $\sum a_i v_i = 0$

$\Rightarrow a_1 = a_2 = \cdots a_n = 0$

now if we add v_{n+1} to A

$v_{n+1} \in V$

then as span $A = V$

$v_{n+1} = \sum b_i v_i$

then as v_{n+1} can be written as form $\sum b_i v_i$ as span $A = V$

$$\{v_1, v_2, \ldots v_n, \underset{\substack{\| \\ \sum_{i=1}^{n} b_i v_i}}{v_{n+1}}\} \text{ is lin dependent}$$

so for $a_i = -b_i$

$\therefore$ A is maninal lin ind subset of V.

① so, B_Y spans Y

② B_Y is lin ind:

$$\sum_{k=0}^{n-j-1} b_k\left(\prod_{i=1}^{j}(x-x_i)\right)x^k = 0, \ \forall x$$

$\swarrow$ poly of degree $(n-j)$

But $\deg\left(\sum_{k=0}^{n-j-1}\left(\prod_{i=1}^{j}(x-x_i)\right)x^k\right) < n-j$

so almost $n-j$ zeros, only if all are zero $\Rightarrow \beta_k = 0$

<u>Note</u>: Better base:

$\{1, t-t_1, (t-t_1)(t-t_2), \ldots\ldots (t-t_1)(\ )(\ldots)\}$

$\swarrow$

$$\prod_{i=1}^{j-1}(t-t_i)$$

($\Leftarrow$) if A is max lin ind subset of V.
then
if v_{n+1} is added to A
A is lin dependent
so $v_{n+1} = \sum_{i=1} a_i v_i$
$\forall v_i \in A$

but as $v_{n+1} \in V$
span $A = \sum_{i=1} a_i v_i = V$
$\therefore$ A is basis of V

5. span$(A) = V$
$A \subseteq V$
A is a basis of V $\Leftrightarrow$ A is a minimal spanning set
① span $A = V$
② A is lin ind
① if we remove a vector
span $A \neq V$

<u>proof</u>:

($\Rightarrow$) span $A = V$ and A is lin ind (as A is basis)
let
$A = \{v_1, v_2, \dots v_n\}$, if we remove v_n (WLG)

but as $v_n \in V$
$\sum a_i v_i \neq v_n$
$\therefore$ span $A \neq V$
$\therefore$ A is a minimal spanning set of V.

($\Leftarrow$) span $A = V$ and A is minimal spanning set
then if we prove A is lin ind then we are done.

$A = \{v_1, v_2, \dots v_n\}$ s.t span $A = V$
if we remove v_n (WLG)

span$\{v_1, v_2, \dots v_{n-1}\} \neq V$

if A was lin dependent
then $\sum_{i=1}^{n} a_i v_i = v_n$
$\therefore$ span $\{v_1, v_2 \dots v_{n-1}\} = V$ $\ast$

$\therefore$ A is lin ind.

$\therefore$ A is basis of V.

Tutorial-3 :

1. $C([0,1]) = \{ f : [0,1] \to \mathbb{C} \mid f \text{ is cont} \}$

$$T : C([0,1]) \to \mathbb{C}$$

$$T(f) = \int_0^1 f(x)x^3 dx, \ \forall f \in C([0,1])$$

for $T(f)$ to be a linear map:

$T(f+g) = T(f) + T(g)$

and $T(\lambda f) = \lambda T(f)$

(i) $T(f+g) = \int_0^1 [f+g](x)\, x^3 dx$

$= \int_0^1 f(x) x^3 dx + \int_0^1 g(x) x^3 dx$

$= T(f) + T(g)$

(ii) $T(\lambda f) = \int_0^1 \lambda f(x) x^3 dx = \lambda \int_0^1 f(x) x^3 dx$

$= \lambda T(f)$

$\therefore T(f)$ is a linear map.

define $C[0,1] \to \mathbb{C}$

by $T(f) = \int_0^1 f(x)x^3 dx \ \forall f \in C[0,1]$

<u>claim</u>: $T(\alpha f + g) = \alpha T(f) + T(g)$

$\forall f, g \in C[0,1]$

and $\alpha \in \mathbb{C}$

$T(\alpha f + g) = \int_0^1 (\alpha f + g)(x) x^3 dx$

$= \alpha \int_0^1 f(x) x^3 dx$

$+ \int_0^1 g(x) x^3 dx$

$\therefore T$ is a linear map

<u>Note</u>: what we are trying to say is $T : V \to W$

$T(V)$ becomes vector subspace of W.

2. $X \leftarrow$ vector space over f

$x \in X$ and $x \neq \bar{0}$

$T(X) = \Sigma \alpha_i$

then

$T(x+y) = \Sigma(x_i + y_i)$

$= \Sigma \alpha_i + \Sigma \beta_i$

$= T(X) + T(Y)$

and $T(\lambda x) = \Sigma(\lambda \alpha_i)$

$= \lambda \Sigma \alpha_i$

$= \lambda T(x)$

$\therefore T(x) \neq 0$ is a linear map.

let X be a vector space over $\mathbb{F}$. Since x_0 is a non-zero vector, $\{x\}$ is a l.I set in X

Extend, $\{x_0\}$ to basis $\mathcal{B}$ of X.

say $\mathcal{B} = \{e_i\}_{i \in I} \cup \{x_0\}$. Define map

$T : X \to \mathbb{F}$ by

$T\left(\sum_{i \in I_0 \subseteq I} c_i e_i + d \cdot x_0\right) = d$

(i) well defined

$x = y \Rightarrow T(x) = T(y)$

since $x = y$

$T(\Sigma c_i e_i + d_1 x_0)$

$= d_1$

$T(\Sigma c_i e_i + d_1 x_0)$

$= d_1$

$d_1 = d_1$

$\therefore T(x) = T(y)$

(ii) $T(\alpha x + y)$

$= T(\Sigma \alpha c_i e_i + (\alpha d_1 + d_2) x_0 + \Sigma \tilde{c}_i e_i)$

$= \alpha d_1 + d_2$

$= \alpha T(x) + T(y)$

3. $C([-1,1]) = \{ f : [-1,1] \to \mathbb{R} \mid f \text{ is cont} \}$

$$P : C([-1,1]) \to C([-1,1])$$

$$(Pf)(x) = \frac{f(x) + f(-x)}{2}, \ \forall x \in [-1,1] \ \& \ f \in C([-1,1])$$

now, $(P(f+g))(x) = \frac{(f+g)(x) + (f+g)(-x)}{2}$

$= \frac{f(x) + f(-x)}{2} + \frac{g(x) + g(-x)}{2}$

$= (Pf)(x) + (Pg)(x)$

$$\text{and } (P(\lambda f))(x) = \frac{\lambda f(x) + \lambda f(-x)}{2}$$

Define $P: C([-1,1]) \to C([-1,1])$ by $(Pf)(x) = \frac{f(x)+f(-x)}{2}$, $x \in [-1,1]$

$$= \lambda\left(\frac{f(x)+f(-x)}{2}\right)$$

$$= \lambda(Pf(x))$$

$$\text{now } (Pf)(x) = \frac{f(x)+f(-x)}{2}$$

we need to show that $P^2 = P$

$\Leftrightarrow P^2(f) = P(f)$ $\forall f \in C[-1,1]$

$$P\left(\frac{f(x)+f(-x)}{2}\right) = \frac{1}{2}\left(\frac{f(x)+f(-x)}{2}\right) + \frac{1}{2}\left(\frac{f(-x)+f(x)}{2}\right)$$

$$= \frac{f(x)+f(-x)}{2}$$

$$= (Pf)(x)$$

$$\therefore P^2 = P$$

$P(P(f)) = P(f)$

$$P(Pf)(x) = \frac{(Pf)(x) + (Pf)(-x)}{2}$$

$$= Pf(x)$$

$$P(P(f)) = P(f)$$

4. $X \leftarrow$ vector space of polynomials

$$T: X \to \mathbb{C}^n$$

$$T(P) = (P(s_1), \dots P(s_n)), \forall P \in X$$

(i) $T(P_1 + P_2) = ((P_1+P_2)(s_1), \dots (P_1+P_2)(s_n))$

$$= T(P_1) + T(P_2)$$

$$T(\lambda P) = (\lambda P(s_1), \dots \lambda P(s_n)) = \lambda T(P)$$

(ii) $T: X \to \mathbb{C}^n$ by

$$T(P) = (P(s_1), \cdots P(s_n)), \forall P \in X$$

$$\text{Basis } X = \{1, x, \dots x^{n-1}\}$$

$$\deg X = n$$

$$P = \sum_{i=0}^{n-1} \alpha_i x^i = \alpha_0 + \alpha_1 x^1 + \cdots \alpha_{n-1} x^{n-1}$$

now, $\mathbb{C}^n$ has deg of n.

as $\deg X = \deg \mathbb{C}^n$

$\Rightarrow X$ is isomorphic to $\mathbb{C}^n$

$\therefore T$ is one-one, onto and well defined

$\therefore N_T = \{0\}$ $\therefore N_T$ is trivial

and T is onto, i.e surjective.

correction: $T: X \to \mathbb{C}^{n+1}$ $s_0, s_1, s_2, \dots \dots s_n$ be distinct

$$T(P) = (P(s_0), P(s_1), \dots \dots P(s_n))$$

(ii) $N_T = \{P \in X \mid T(P) = (0, \dots 0)\}$

$P(s_0) = 0 \dots P(s_n) = 0$

P of degree $n+1$ then by fundamental theorm of algebra it has $n+1$ root

$*$ $\therefore P \notin N_T$

$\therefore N_T = \{0\}$

$\therefore \dim N_T = 0$

N_T is trivial

By rank-nullity theorm

$\dim N_T + \dim R_T = \dim X$

$\dim R_T + 0 = n$

$\dim R_T = n$

but $\dim \mathbb{C}^{n+1} = n+1$

$\therefore$ not surjective

Example: $v \in \mathbb{C}^{n+1}$ s.t $T(P) \neq v$

$P(s_n) = 1$

(iii) $\int_0^1 P(x)\,dx = m_1 P(t_1) + \cdots m_n P(t_n), \quad \forall P \in X$
$t_i \in [0,1]$

we know $\dim(X) = n$
It follows that $\dim(X') = n$
dual space

Proof: $\int_0^1 \sum_{i=0}^{n-1} \alpha_i x^i dx = \alpha_0 + \alpha_1 \int_0^1 x\,dx + \alpha_2 \int_0^1 x^2 dx + \cdots \alpha_{n-1} \int_0^1 x^{n-1} dx$

$$= \alpha_0 + \frac{\alpha_1}{2} + \frac{\alpha_2}{3} + \cdots \frac{\alpha_{n-1}}{n}$$

$X \ni P \to \int_0^1 P(x)dx \in \mathbb{C}$

$\psi(P) = \int_0^1 P(x)dx$

$\psi : X \longrightarrow \mathbb{C}$

Linear $\Rightarrow \psi_n \in X'$

now $m_1 P(t_1) + \cdots m_n P(t_n)$

$$= m_1(\alpha_0 + \alpha_1 t_1 + \cdots \alpha_{n-1} t_1^{n-1})$$
$$+ m_2(\alpha_0 + \alpha_1 t_2 + \cdots \alpha_{n-1} t_2^{n-1})$$
$$+ m_3(\alpha_0 + \alpha_1 t_3 + \cdots \alpha_{n-1} t_3^{n-1})$$
$$\vdots$$
$$\ddots\ m_n(\alpha_0 + \alpha_1 t_n + \cdots \alpha_{n-1} t_n^{n-1})$$

Since $\psi = \sum_{j=1}^{n} \alpha_j \ell_j$ $\forall j \in \{1,\dots n\}$

$t_j \in [0,1]$

$\delta_{t_j} : X \to \mathbb{C}$

$P \to P(t_j)$

prove $\{\delta_{t_j}\}_{j=1}^{n}$ this is ind. in dual space

$$= \alpha_0(m_1 + m_2 + \cdots m_n)$$
$$+ \alpha_1(m_1 t_1 + m_2 t_2 + \cdots m_n t_n)$$
$$+ \alpha_2(m_1 t_1^2 + m_2 t_2^2 + \cdots m_n t_n^2)$$
$$\vdots$$
$$\alpha_{n-1}(m_1 t_1^{n-1} + m_2 t_2^{n-1} + \cdots m_n t_n^{n-1})$$

$\{\delta_{t_j}\}_{j=1}^{n}$ Basis

$\Rightarrow \exists\, m_j \quad j = (1,2,\dots n)$
s.t. $\psi = \sum_{j=1}^{n} m_j \delta_{t_j}$

$\Rightarrow \psi(P) = \sum m_j \delta_{t_j}(P) \quad \forall P \in X$
$= \int_0^1 P(x)dx$

for distinct $t_i \in [0,1]$

Note: there are n equations, and n variables m_i.

$$\therefore \begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ \vdots & & & \\ t_1^{n-1} & \cdots & \cdots & t_n^{n-1} \end{bmatrix}$$

if invertible then this exist

for $t_i \in (0,1)$
$1 \neq t_1 \neq t_1^2 \cdots$
and $t_i \neq t_j$ for $i \neq j$
every colum is differt
$\Rightarrow$ Basis
$\therefore$ the matrix is invertible.

tutorial-4:

1. $X \leftarrow$ finite dim. vector space

$M \in \mathcal{L}(X,X)$ $N \in \mathcal{L}(X,X)$

linear map

similar if

$\exists$ invertible $S \in \mathcal{L}(X,X)$ s.t
$M = SNS^{-1}$

(i) $M \sim N$ as $M = SNS^{-1}$ $M \sim N$ is M and N is similar

① $M \sim M$ as for $S = I \in \mathcal{L}(X,X)$
$M = IMI^{-1}$ (Reflexive)

② $M \sim N$ then
for $S' = S^{-1}$
$N = S^{-1}MS$
$= S'MS'^{-1}$ (symmetric)

$\therefore N \sim M$

③ if $M \sim N$ and $N \sim P$
then
$M = S_1 N S_1^{-1}$
$N = S_2 P S_2^{-1}$

then $M = S_1 (S_2 P S_2^{-1}) S_1^{-1}$
$= (S_1 S_2) P (S_1 S_2)^{-1}$

$S = S_1 S_2$
as S_1, S_2 invertible
S is also invertible

$\therefore M \sim P$ (transitive)

(ii) wlog N is invertible. i.e
$\exists B$ s.t
$NB = BN = I$
and $B = N^{-1}$

now, $M = SNS^{-1}$
$MN = SNS^{-1}N$
and $NM = NSNS^{-1}$

now $MN = N^{-1}NSNS^{-1}N = N^{-1}(NSNS^{-1})N$
$MN = N^{-1}(NM)N$
$\therefore$ MN and NM are similar for
$S' = N$

(iii) $A = PBP^{-1}$
then $Ax = 0 \Leftrightarrow PBP^{-1}x = 0$
$\Leftrightarrow BP^{-1}x = 0$
$\Leftrightarrow B(P^{-1}x) = 0$
here $N_{P^{-1}} = \{0\}$
so the $\forall x$ s.t $Ax = 0$
and $B(P^{-1}x) = 0$
will have the same basis
as if Basis $= \{x_1, \dots x_n\}$
Basis for $P^{-1}x = \{P^{-1}(x_1), \dots P^{-1}(x_n)\}$
$\therefore \dim N_A = \dim N_B$
$\Rightarrow \dim R_A = \dim R_B$

$\mathcal{L}(X,X) \rightarrow \mathcal{L}(X,X)$
$A \rightarrow SA$
$y \in N_A \Rightarrow Ay = 0$
$\Rightarrow SAy = 0$
$\Rightarrow y \in N_{SA} \Rightarrow N_A \subseteq N_{SA}$
similarly if $y \in N_{SA}$
$\Rightarrow SAy = 0$
$\Rightarrow Ay = 0$
$\Rightarrow y \in N_A$
$\therefore N_A = N_{SA}$

now $A \rightarrow AS$
$N_A \rightarrow N_{AS}$
$\downarrow$ $\downarrow$

$\{e_i\}_{i=1}^{m}$ $\{S(e_i)\}_{i=1}^{m}$ Bijective

$ASw=0$
$\Rightarrow w \in NA$
$w = \sum c_i e_i$
$w = \sum c_i (S^{-1} e_i)$
$\therefore$ span is proved.

for basis : indipendent set

$\{S^{-1} e_i\}_{i=1}^{m}$ is injective

$l_i : X \to \mathbb{F}$

$L = \sum \beta_j l_j = \sum L(x_j) l_j$

2. $X \leftarrow$ vector space over F

$x = \{x_1, \dots x_n\}$

Basis $\forall\ 1 \le i \le n$

$l_i : X \longrightarrow F$

$l_i(x_j) = \delta_{i,j}\ \forall\ j = 1, \dots n$

$$\delta_{i,j} = \begin{cases} 1_F ; & i = j \\ 0 ; & i \neq j \end{cases}$$

$l = \{l_1, \dots l_n\}$ Basis of X'

$l_i \in X'$

as l_i is a linear function

$$\pi : \alpha(X, F) \ (X') \longrightarrow \underbrace{F \times F \times \dots F}_{n \text{ times}}$$

$$\Gamma \longrightarrow (\Gamma(x_1), \dots \Gamma(x_n))$$

Here we have already proved that this is isomorphic

Now to prove that

$l = \{l_1, l_2, \dots l_n\}$ is the basis

① it spans $\alpha(X, F)$

Now as $\pi(\Gamma(x)) = (\Gamma(x_1), \Gamma(x_2), \dots \Gamma(x_n))$

$\pi(l_i(x)) = (l_i(x_1), \dots l_i(x_i), \dots l_i(x_n))$
$= (0, 0, \dots 0, 1, 0, \dots 0)$ — i^{th} place

then $\pi(l_i(x))$ for all $i \in \{1, 2, \dots n\}$ is like

$\pi(l_1(x)) = (1, 0, \dots 0)$
$\pi(l_2(x)) = (0, 1, \dots 0)$
$\vdots$
$\pi(l_n(x)) = (0, 0, \dots 1)$

now, as $\sum \alpha_i \pi(l_i(x)) = \pi(\sum \alpha_i l_i(x))$
$= \sum \alpha_i f_i$
$\forall \alpha \in \mathbb{R}$

this will be F^n

or it spans F.

as π is isomorphic

$\sum \alpha_i l_i(x)$ spans $\alpha(X, F)$

Also as f_i's are lin ind, $l_i(x)$ are linearly ind (from prev q)

$\therefore$ l is the basis

3. $X \leftarrow$ n-dim vetor space over F.

(i) ℓ be a non-zero linear fuction

$$\ell : X \longrightarrow F$$

$$\dim(X) = \dim(R_\ell) + \dim(N_\ell)$$

as $\ell \neq 0$ $\dim(N\ell) \neq \dim(X)$

$\therefore \dim(N\ell) < \dim(X)$

also $\dim(R\ell) < \dim(X)$

$\dim(N\ell) \geq 0$

$\dim(R\ell) > 0$

$$0 \leq \dim(N_\ell) < \dim(X)$$

See here as ℓ is a linear map non-zero $\ell : X \to F$

$\exists\, x \in X$ s.t $(x \neq 0)$

$\ell(x) \neq 0$

Also $\ell(x) = F \Rightarrow v$

$\forall\, x \in X$ is onto map

$x_0 = \ell(x)$

$\ell(y) = v$

$v = \frac{x_0}{x_0} \cdot v = x_0 \left(\frac{v}{x_0}\right)$

$\ell(y) = x_0 \cdot \frac{v}{x_0} = \ell(x) \frac{v}{x_0}$

$\ell(y) = \ell\left(x \frac{v}{x_0}\right)$

$\therefore \forall\, v \in F$

$\exists\, x \frac{v}{x_0}$ s.t

$\ell\left(x \frac{v}{x_0}\right) = v$ $\therefore$ onto

This involves isomorphism $\tilde{\ell} : \frac{X}{N\ell} \to F$

isomorphism As this is onto

$\left|\frac{X}{N\ell}\right| \cong F$

$\dim\left(\frac{X}{N\ell}\right) = 1$

$\dim(X) - \dim(N\ell) = 1$

$\dim(N\ell) = \dim(X) - 1$

(ii) To prove: $f : X \longrightarrow F$

$f = \alpha \ell \iff N_\ell \subseteq N_f$

proof : $(\Rightarrow)$ given $f = \alpha \ell$

then if $x \in N_\ell$

then $\ell(x) = 0$

$\Rightarrow \alpha \ell(x) = 0$

$\Rightarrow f(x) = 0$

$\therefore x \in N_f$

$\therefore N_\ell \subseteq N_f$

$(\Leftarrow)$ $f : X \to F$

$\ell : X \to F$

and α be some number in F

then

if $N\ell \subseteq N_f$

then

$f(N\ell) = 0$

also as f is a linear map

$f(\alpha' x + y) = \alpha' f(x) + f(y)$

and $\ell(\alpha' x + y) = \alpha' \ell(x) + \ell(y)$

If f is zero, then $\alpha = 0$ (trivial case)

if $f \neq 0$ (Non-zero function) then

Note that

$\tilde{f} : \frac{X}{N\ell} \to F$, $[x]$

$x + N\ell \to f(x)$

Well defined

$x - y \in N\ell$

$\Rightarrow x - y \in N_f$

$\Rightarrow f(x) = f(y)$

$\tilde{\ell} : \frac{X}{N\ell} \to F$, $[x] \hookrightarrow$

$\tilde{f}([x]) = f(x) \neq 0$

$\tilde{\ell}([x]) = \ell(x) \neq 0$

$\frac{f(x)}{\ell(x)} = \alpha$

now as $f(x) = g(x) \cdot \ell(x)$

for some fuction

as f is a linear map

$f(\alpha x + y) = g(\alpha x + y)\ \ell(\alpha x + y)$

$= (\alpha g(x) + g(y))\ (\alpha \ell(x) + \ell(y))$

then

$\alpha g(x) + g(y) \in F$

If not then f is not a linear map

$\exists\, \alpha' \in F$

$f(x) = \alpha' \ell(x)$

$f = \alpha \ell$

$\Leftrightarrow f(y) = \alpha \ell(y)$

$= \frac{f(x)}{\ell(x)} \ell(y)$

See this prof soln

here

$f(y) = \tilde{f}([y])$ $y \neq N\ell$

$= \tilde{f}(\lambda [x])$

$= \lambda f(x)$

$\frac{f(x)}{\ell(x)} \ell(y) = \frac{f(x)}{\ell(y)} \tilde{\ell}([y])$

$= \frac{f(x)}{\ell(x)} \lambda \cancel{\ell(x)}$

$= \lambda f(x) = f(y)$

(iii) V be a subspace of X $\dim(V) = n-1$

$\dim X = n$

$g : X \to F$

$N_g = V$

$g : X \to F$

s.t $N_g = V$

basis $V = \{v_1, v_2, \ldots, v_{n-1}\}$

also basis of X

can be made as $\{v_1, \ldots, v_{n-1}, x_n\}$

(ii) $(\Leftarrow)$

assume $N_\ell \subseteq N_f$

since $\dim(N\ell) = n-1$

as non-zero let

if $f = 0$

$N_f : X \to 0$

$N_f = X$

trivial case ↑ as $f=0$ for $\alpha=0$ $f=0$

$N_f = X \supseteq N_\ell$

if $\dim N_f = \dim N_\ell$ then $N_f \cong N_\ell$

$\tilde{f} : \frac{X}{N_f} \to \mathbb{C}$

$x + N_f \mapsto f(x)$

$\tilde{\ell} : \frac{X}{N_\ell} \to \mathbb{C}$

$\ell + N_\ell \mapsto \ell(x)$

$\tilde{f} : \frac{X}{N_\ell} \to \mathbb{C}$ $x + N_\ell \mapsto f(x)$

now, $g(x) = g(\sum \alpha_i v_i + \alpha_n x_n)$
$= \alpha_n$

so my function is defined as this.

① well defined as:

$\sum \alpha_i v_i + \alpha_n x_n = \sum \beta_i v_i + \beta_n x_n \quad (\alpha_n = \beta_n)$

$\Rightarrow g(\sum \alpha_i v_i + \alpha_n x_n) = g(\sum \beta_i v_i + \beta_n x_n)$

② <u>linear map</u> :

$g(\alpha x + y) = g(\alpha \sum \alpha_i v_i + \alpha(\alpha_n x_n)) + \sum \alpha_i' v_i + \alpha_n' x_n$

$= \alpha \alpha_n + \alpha_n'$

$= \alpha g(\ ) + g(\ )$

$N_g = V$

check if this is well defined or not

$x + N_\ell = y + N_\ell \Leftrightarrow x - y \in N_\ell \subseteq N_f$

$\Rightarrow f(x) = f(y)$

Also isomorphism as $\text{Basis}(N_\ell) = \{x_1 \ldots x_{n-1}\}$

$\text{Basis}(X) = \{x_1 \ldots x_n\}$

if $f(x_n + N_\ell) = 0$

$\Rightarrow f(x_n) = 0$

$\Rightarrow x_n \in N_f$

Note $\tilde{\ell} : \frac{X}{N_\ell} \to \mathbb{C}$ $x_n + N_\ell$ Basis

$\tilde{\ell}(x_n + N_\ell) \neq 0$

$\tilde{\ell} = \alpha \tilde{f}$ on $\frac{X}{N_\ell}$

$\Rightarrow \ell(x) = \alpha f(x), \forall x \in X$

(iv) $g : \mathbb{R}^2 \to \mathbb{R}$

N_g is line $ax + by = 0$, $a, b \in \mathbb{R}$

$g(x, y) = ax + by$

$g\begin{pmatrix} x \\ y \end{pmatrix} = ax + by$

then $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \Rightarrow ax_1 = ax_2$ & $by_1 = by_2$

$\Rightarrow ax_1 + by_1 = ax_2 + by_2$

$\therefore$ well defined

now, $g\left(\begin{pmatrix} x \\ y \end{pmatrix} + \alpha\begin{pmatrix} x' \\ y' \end{pmatrix}\right) = g\begin{pmatrix} x + \alpha x' \\ y + \alpha y' \end{pmatrix} = a(x + \alpha x') + b(y + \alpha y')$

$= ax + by + \alpha a x' + \alpha b y'$

$= g\begin{pmatrix} x \\ y \end{pmatrix} + \alpha g\begin{pmatrix} x' \\ y' \end{pmatrix}$

(iii) let $V \subseteq X$ with $\dim(V) = n-1$

$\frac{X}{V} \xrightarrow{f_1} \mathbb{C}$

$f_1(\underbrace{\beta[x_0]}_{\text{general element in the Quotient space}}) = \beta\alpha$, $\alpha = f_1([x_0])$

$X \xrightarrow{\pi} \frac{X}{V} \xrightarrow{f_1} \mathbb{F}$

$x \mapsto x + V$

$f := f_1 \circ \pi$

$f(x_0) = f_1(\pi(x_0)) = \alpha \neq 0$

now $N_g = V$

$X \supseteq$ prove $\to \ker(f_1 \circ \pi) = \ker(\pi) = V$

almost $n-1$ to prove.

$f_1 \leftarrow$ injective

$\pi \leftarrow$ surjective

$f_1(\pi(y)) = 0$

$\Rightarrow \pi(y) = 0$

$\Rightarrow y + V = 0 \Rightarrow y \in V$

4. $T(x_1, x_2) = (-x_2, x_1)$

(i) $T : \mathbb{R}^2 \to \mathbb{R}^2$

$\therefore T$ is a 2×2 matrix

now $T(e_1) = (0, 1)$

$T(e_2) = (-1, 0)$

Now $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

then $T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow T(I) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\therefore Ax = T(x)$

s.t $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$

(ii) $T(x_l) = \sum_{i=1}^{n} A_{il} x_i$

wrt x_l bases

$$T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix} = A_{11}\begin{pmatrix}1\\2\end{pmatrix} + A_{21}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} = A_{12}\begin{pmatrix}1\\2\end{pmatrix} + A_{22}\begin{pmatrix}1\\-1\end{pmatrix}$$

$-2 = A_{11} + A_{21}$

$1 = 2A_{11} - A_{21}$

$\boxed{-\frac{1}{3} = A_{11}}$

$1 = -\frac{2}{3} - A_{21}$

$\boxed{A_{21} = -\frac{5}{3}}$

$-2 = 2A_{12} + 2A_{22}$

$1 = 2A_{12} - A_{22}$

$-3 = 3A_{22}$

$\boxed{A_{22} = -1}$

$\boxed{A_{12} = 0}$

doubt (see how to do this properly)

$$\therefore A = \begin{bmatrix} -1/3 & 0 \\ -5/3 & -1 \end{bmatrix}$$

Tutorial-5:

1. $\det(A) = D(A_{*1}, A_{*2}, \ldots A_{*n})$

now

$D(\alpha A_{*1}, A_{*2}, \ldots A_{*n})$

$= \alpha D(A_{*1}, \ldots A_{*n})$

Similary $\alpha^n D(A_{*1}, \ldots A_{*n}) = D(\alpha A_{*1}, \alpha A_{*2}, \ldots \alpha A_{*n})$

$\alpha^n \det(A) = \det(\alpha A)$

2. $A^2 = 0$

now $\det(cI - A)$

Note A is 2×2, so $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

s.t $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & d^2+cb \end{bmatrix} = 0$

also $\det(\alpha I - A) = \det\left(\begin{bmatrix} \alpha-a & -b \\ -c & \alpha-d \end{bmatrix}\right)$

$\det(A) = \sum_{\sigma \in S_2} \operatorname{sgn}(\sigma) a_{\sigma(1)1} a_{\sigma(2)2}$ for 2×2 matrix

$= a_{11}a_{22} - a_{12}a_{21}$

$(I - A = (c - a_{11})(c - a_{22}) - a_{12}a_{21}$

$= c^2 - ca_{22} - ca_{11} + a_{11}a_{22} - a_{12}a_{21}$

$= c^2 - ca_{22} - ca_{11}$

as $a_{11} = -a_{21}$

$= c^2$

$= (\alpha - a)(\alpha - d) - (-c)(-b)$

$= \alpha^2 - \alpha d - \alpha a + ad - cb$

$= \alpha^2 - \alpha(a+d) + ad - cb$

$a^2 = d^2 = -bc$

$a + d = 0$

or $c = 0, b = 0$

as if $a + d = 0$

then

$\alpha^2 - \alpha(0) + (a)(\cancel{-a}) \cancel{+a^2}$

$= \alpha^2$

if $c = 0, b = 0$

then everything zero

$\therefore \alpha^2$ only

$\therefore \det(\alpha I - A) = \alpha^2$

or $\det(cI - A) = c^2$

3. As $\det(A^T) = \det(A)$

here $\det(A^T) = \det(-A) = (-1)^n \det(A) = \det(A)$

$\Rightarrow \det(A) = 0$

as n is odd

4. $AA^T = I$

$\searrow$ orthogonal

$\det(AA^T) = 1$

$\Rightarrow \det(A)\det(A^T) = 1$

$\Rightarrow \det^2(A) = 1$

$\Rightarrow \det(A) = \pm 1$

if $\det(A) = -1$ then let $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ also $AA^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

5. S_n permutation group of n-symbols

$$\mathrm{sgn}: S_n \to \{\pm 1\}$$
$$\sigma \mapsto \mathrm{sgn}(\sigma)$$

now let $N(\sigma_1)$ be number of 2-cycles in σ_1
and $N(\sigma_2)$ be number of 2-cycles in σ_2

<u>Note</u> not unique and for

$$N(\sigma_1\sigma_2) = N(\sigma_1) + N(\sigma_2) \mod(2)$$

or $$(-1)^{N(\sigma_1\sigma_2)} = (-1)^{N(\sigma_1)+N(\sigma_2)}$$

$$\therefore \mathrm{sgn}(\sigma_1 \circ \sigma_2) = \mathrm{sgn}(\sigma_1)\,\mathrm{sgn}(\sigma_2)$$
$$\forall\, \sigma_1, \sigma_2 \in S_n$$

Tutorial 6:

1. $T: \mathbb{F}^n \to \mathbb{F}^n$ (linear map)

$\det(T(\alpha_1), \dots T(\alpha_n)) = c \det(\alpha_1, \dots \alpha_n)$

$c = \det(T(e_1), T(e_2), \dots T(e_n))$

as

$$\det(T(\alpha_1), \dots T(\alpha_n))$$
$$= \det \begin{bmatrix} T\alpha_1 & T\alpha_2 & \dots & T\alpha_n \\ | & | & & | \end{bmatrix}$$
$$= \det \begin{bmatrix} T \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \end{bmatrix}$$
$$= \det \begin{bmatrix} T \end{bmatrix} \det \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \end{bmatrix} \quad (\because \det(AB) = \det(A)\det(B))$$
$$= \det(T(e_1), T(e_2), \dots T(e_n)) \det(\alpha_1, \alpha_2, \dots \alpha_n)$$

2. $A, B \in M_{n \times n}(\mathbb{C})$, $\lambda, \beta \in \mathbb{C}$ s.t

(i) λ is eigenvalue of A, β is eigenvalue of B

or $Ax = \lambda x$ (for some $x \in \mathbb{C}^n$)

$By = \lambda y$ (for some $y \in \mathbb{C}^n$)

$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}$

$3\lambda + \lambda^2 - (-2) = 0$

$\lambda^2 + 3\lambda + 2 =$

$\lambda^2 + \lambda + 2\lambda + 2 = 0$

$\lambda(\lambda+1) + 2(\lambda+1) = 0$

$\Rightarrow \lambda = -2, -1$

now, $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ $\quad \beta = 5, -2$

then $A + B = \begin{bmatrix} 1 & 5 \\ 1 & -1 \end{bmatrix}$

$(A+B) - \lambda'(I) = \begin{bmatrix} 1-\lambda' & 5 \\ 1 & -1-\lambda' \end{bmatrix}$

$(1-\lambda')(-1-\lambda') - 5 = 0$

$-(1+\lambda')(1-\lambda') = 5$

$-(1-\lambda'^2) = 5$

$\lambda'^2 - 1 = 5$

$\lambda'^2 = 6$

$\lambda' = \pm\sqrt{6} \neq \lambda + \beta$

(ii) $AB = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -11 & -14 \end{bmatrix}$

$(3-\lambda)(-14-\lambda) - (-11)(2) = 0$

$-42 - \cancel{3\lambda} + \cancel{14\lambda} + \lambda^2 + 22 = 0$

$\lambda^2 + 11\lambda - 20 = 0$

$\lambda = \dfrac{-11 \pm \sqrt{121+80}}{2}$

$= \dfrac{-11 \pm \sqrt{201}}{2}$ Not perfect square

3. $\lambda \leftarrow$ eigvalue of $A \in M_{n \times n}(\mathbb{F})$

$\lambda^n \leftarrow$ eigvalue of A^n

for some $x \neq 0$

$Ax = \lambda x$

$\Rightarrow A^2 x = A(Ax) = A(\lambda x)$

$= \lambda(Ax)$

$= \lambda . \lambda x$

$= \lambda^2 x$

for $n=2$ true

say $n=k$ true

then, $A^k x = \lambda^k x$

for $A^{k+1} x = A(A^k x)$

$= A(\lambda^k x)$

$= \lambda^k (Ax)$

$= \lambda^{k+1} x$

$\therefore$ By induction

$\lambda^n x = A^n x$

or λ^n is eigvalues of A^n

$P \rightarrow$ polynomial $\quad P(A) = \sum_{i=0}^{n} \alpha_i A^i$

for $A^0 = I$

$P(A)x = \sum \alpha_i A^i x$

$= \sum \alpha_i \lambda^i x$

$P(A)x = P(\lambda)x$

or $P(\lambda)$ is an eigvalue of $P(A)$

4. $A \in M_{n \times n}(\mathbb{C})$, nilpotent matrix

$A^r = 0$ $r \in \mathbb{N}$

$A^r = 0$

say λ is an eignvale of A
then

$A x = \lambda x$

$\lambda \neq 0, \quad x \neq 0$

$\Rightarrow A^r x = \lambda^r x$

$\Rightarrow 0 = \lambda^r x$

for this $\lambda \neq 0, x \neq 0$
not possible

so <u>no</u> eignvalues.

5. $M \sim N$

$S M S^{-1} = N$

$\det(N - xI) = \det(S N S^{-1} - xI)$

$= \det(M - xI)$

$\chi_N(x) = \chi_M(x)$

$\therefore$ same eignvalues

Tutorial 7:

1. similar matrices

$M \sim N$

$\exists S \in M_{n\times n}(\mathbb{C})$ s.t

$SMS^{-1} = N$

$\chi_N(s) = \det(N - sI)$

$\chi_N(t) = \det(SS^{-1})\det(N - tI)$

$= \det(SNS^{-1} - tSS^{-1}I)$

$= \det(M - tI)$

$\therefore \chi_N(t) = \chi_M(t)$

$\therefore$ same eigenvalues.

2. as $\det T = \det T^t$

$\det(T + It)^t = \det(T^t + It)$

or

$$\chi_{T^t} = \chi_T$$

3. $T \in M_{n\times n}(\mathbb{C})$

n distinct eigenvalues $\{\lambda_1, \ldots \lambda_n\}$

first generally:

then as $T \sim \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, $\exists S \in M_{n\times n}(\mathbb{C})$ s.t

$$STS^{-1} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$\Rightarrow TS = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} S$$

$\Rightarrow TS_{*i} = \lambda_i S_{*i}$

as S is invertible

$\{S_{*1}, \ldots S_{*n}\}$ form a basis of $\mathbb{C}^n$

or: $h = \sum_{i=1}^{n} \alpha_i S_{*i}$

this means that $T^m h = T^m\left(\sum \alpha_i S_{*i}\right)$

$= \sum \alpha_i T^m S_{*i}$

$= \sum \alpha_i \lambda_i^m S_{*i}$

(i) if $|\lambda_i| < 1 \ \forall \ i \in \{1, 2, \ldots n\}$

as $m \to \infty$

$|\lambda_i|^m \to 0$

or $\sum \alpha_i \lambda_i^m S_{*i} \longrightarrow 0$

$\therefore T^m h \to 0$

as $m \to \infty$

$\forall u \in \mathbb{C}^n$

(ii) for $\forall r > 0$, $\exists n_0 = \max\left\{ \left\lfloor \log_{|\lambda_i|} \left| \frac{r}{n|\alpha_i||S_{ji}|} \right| \right\rfloor + 1 \right\}$

s.t a row j:

$$|\Sigma \alpha_i \lambda_i^m s_{ji}| = \Sigma |\alpha_i| |\lambda_i|^m |s_{ji}|$$

$$> \Sigma \cancel{|\alpha_i|} \left(\frac{r}{(n)(\cancel{|\alpha_i|}|\cancel{s_{ji}}|)} \right) \cancel{|s_{ji}|}$$

$$= \Sigma \frac{r}{n} = r$$

$$|\Sigma \alpha_i \lambda_i^m s_{ji}| > r$$

or elep row $\to \infty$

$\therefore$ $T^m n$ as $m \to \infty$

and $n \neq 0 \in \mathbb{C}^n$

$\longrightarrow \infty$

4. $Tf = \lambda f$ for T to have eigenvalues $(\neq 0)$

$$\int_0^x f(t)\,dt = \lambda f(x)$$

Note: $f(0) = 0$ —— ①

$$\Rightarrow f(x) = \lambda f'(x)$$

now, $\int \frac{1}{\lambda} = \int \frac{f'(x)}{f(x)}$

$$\frac{1}{\lambda} x + c = \ln |f(x)|$$

$$\Rightarrow f(x) = Ce^{x/\lambda}$$

Now $f(0) = 0$

$\Rightarrow$ $ce^{x/\lambda} = 0$

$\forall x \in \mathbb{R}$

as $\lambda \neq 0 \Rightarrow c = 0$

but for $c = 0$

$f \equiv 0$ (eigenvector should be <u>non</u> zero)

$\therefore$ No eigenvalue of T.

5. $A, B \in M_{n \times n}(F)$

$$C = (I - AB)^{-1}$$
$$C(I - AB) = I$$
$$C - CAB = I$$
$$C = I + CAB \quad \text{—— ①}$$

now, $[I + BCA][I - BA]$

$$= I - BA + BCA - BCABA$$
$$= I - BA + BCA(I - BA)$$
$$= (I - BA)(I + BCA) \quad \text{—— ②}$$

also $I - BA + BCA - BCABA = I - \cancel{BA} + \cancel{B(C)A}$

$-\cancel{B(C-I)A}$ from ①

$$= I$$

Tutorial sheet 8:

1. $A \in M_{2\times2}(\mathbb{C})$ rank 1 matrix
or
$$A = \begin{bmatrix} a & ta \\ b & tb \end{bmatrix}$$

now,
$$\det(A - \lambda I) = (a-\lambda)(tb-\lambda) - tab$$
$$= \cancel{atb} - a\lambda - tb\lambda + \lambda^2 - \cancel{tab}$$
$$= 0$$
$$\lambda^2 = a\lambda + tb\lambda$$
$$\Rightarrow \lambda = 0 \text{ or } \lambda = a + tb$$

as λ has 2 distinct values $\Rightarrow A \sim$ diagonal matrix
$\Rightarrow A$ is diagonalisable

2.
$$T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix} \rightarrow \text{Real matrix}$$

$$\chi_T(t) = \det \begin{bmatrix} t & 0 & 0 & 0 \\ -a & t & 0 & 0 \\ 0 & -b & t & 0 \\ 0 & 0 & -c & t \end{bmatrix}$$
$$= t^4$$

now $t^4 = 0$
$\Rightarrow \lambda = 0$ is only eigenvalue of T

or by the theorem
minimal poly of T
$$P(t) = (t-0)^1$$
$$\Rightarrow P(T) = T = 0$$
$$\Rightarrow a = b = c = 0$$

3. A is 2×2 symmetric i.e $A^T = A$

To prove : A is diagonisable

Proof : As $A^T = A$

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = A \quad \text{as } A = A^T$$

now

$$\chi_A(t) = \left| \begin{bmatrix} a-t & b \\ b & d-t \end{bmatrix} \right| = ad - at - dt + t^2 - b^2$$
$$= 0$$

$$t^2 - (a+d)t + ad - b^2 = 0$$

$$\lambda = \frac{(a+d) \pm \sqrt{a^2 + d^2 + 2ad - 4ad + 4b^2}}{2}$$

$$\lambda = \frac{a+d \pm \sqrt{(a-d)^2 + b^2}}{2}$$

$\therefore$ diff $\lambda \Rightarrow A \sim$ diagonal matrix

4. $A \in M_{n \times n}(\mathbb{C})$ s.t
$A^k = 0$ for some $k \in \mathbb{N}$

or as $A^k = 0$ we have a polynomial
$f(x) = x^k$
s.t $f(A) = A^k = 0$
or
$f(x) \in \langle p \rangle$
$\hookrightarrow$ minimal poly of A

then $p(x) \mid f(x)$
$p(x) \mid x^k \Rightarrow p(x) = x^m$
where $m \leq k$

as roots of minimal poly = eigvalues
we have

$P(x) = x^m = 0$
$\Rightarrow \underline{x = 0}$

or $x = 0$ is the eigvalue of A

$Ax = 0$
for some $x \neq 0$

$\Rightarrow \chi_A(x) = x^n$
as deg $\chi_A = n$
& also
roots of $\chi_A(x)$
= roots
of $p(x)$

$\Rightarrow$ By caley namilton theorm
$\chi_A(A) = 0$
$\Rightarrow A^n = 0$

5. $N \in M_{2\times 2}(\mathbb{C})$ s.t $N^2 = 0$ or $N \sim \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

assume that $N \neq 0$ then
$\exists x \neq 0$ in $\mathbb{C}^2$ s.t
$Nx \neq 0$

$\{x, Nx\}$ basis for $\mathbb{C}^2$
or N is cyclic nilpotent $\therefore N \sim \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

as $\quad Tx = 0\cdot x + (1)(Tx)$
$T^2 x = 0\cdot x + (0)\, Tx$

or $\quad T \sim \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

6. $T : M_{n\times n}(F) \longrightarrow M_{n\times n}(F)$ defined by
$$B \longmapsto \underset{\text{fixed}}{\underset{\uparrow}{A}}B$$

if $\lambda \in \sigma(T)$ then
$\exists\, C \in M_{n\times n}(F) \neq 0$
s.t
$T(C) = AC = \lambda C$

or $AC = \lambda C$

$(A - \lambda I)C = 0$

as $C \neq 0 \Rightarrow A - \lambda I = 0$
$\Rightarrow \lambda \in \sigma(A)$

$\therefore \forall\, \lambda \in \sigma(T) \Rightarrow \lambda \in \sigma(A)$

now if $\lambda \in \sigma(A)$ then

$A - \lambda I = 0$
$\Rightarrow (A - \lambda I)C = 0$ for non-zero C

$\Rightarrow T(C) - \lambda(C) = 0$
$\Rightarrow TC = \lambda C$
$\Rightarrow \lambda \in \sigma(T)$

so $\sigma(T) = \sigma(A)$
as eigvalues same $\Rightarrow$ roots of minimal polynomial of T
= roots of minimal polynomial of A

as roots same & minimal poly both

$\Rightarrow P_T = P_A$

Tutorial- 9:

1.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ $\chi_A(t) = -(t-2)(t-1)^2$

now

$\text{Nullity}(A-I) = 2$

or

$GM(A-I) = 2$

$\therefore \begin{bmatrix} (1) & & \\ & (1) & \\ & & (2) \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\chi_A(t) = (t-1)^4$

now nullity $\underbrace{\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A-I} = 1 = GM(A)$

$\therefore J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ -6 & -5 & 3 \end{bmatrix}$ $\chi_A(\lambda) = -(\lambda-2)(\lambda^2-2\lambda-2)$

$= -(\lambda-2)\left(\lambda - \left(\frac{2+\sqrt{4+8}}{2}\right)\right)\left(\lambda - \left(\frac{2-\sqrt{12}}{2}\right)\right)$

$= -(\lambda-2)(\lambda-1-\sqrt{3})(\lambda-1+\sqrt{3})$

$P_A(\lambda) = (\lambda-2)(\lambda-1-\sqrt{3})(\lambda-1+\sqrt{3})$

$J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1+\sqrt{3} & 0 \\ 0 & 0 & 1-\sqrt{3} \end{bmatrix}$

2. $\chi A(\lambda) = (\lambda)^n$

$PA(\lambda) = (\lambda)^n$

$GM(A) = 1$

$J = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix}_{n \times n}$

x_1 s.t $Ax_1 = 0$

x_2 s.t

$x_1 = Ax_2$

$x \neq 0$

then $T^{n-1}x$ as $x_1 = T^{n-1}x$

$T^{n-2}x$ $x_2 = T^{n-2}x$

$\vdots$ $\vdots$ $\forall x \neq 0$

x $x_n = x$

3. $J_r(\lambda)$ = Jordan block of λ with $r \times r$

$$J_r(\lambda) = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}_{r \times r}$$

$$T \sim \begin{bmatrix} J_{n_1}(\alpha) & 0 & 0 & 0 \\ 0 & J_{n_2}(\alpha) & 0 & 0 \\ 0 & 0 & J_{m_1}(\beta) & 0 \\ 0 & 0 & 0 & J_{m_2}(\beta) \end{bmatrix}$$

then minimal polynomial of

$$P_T(\lambda) = (\lambda - \alpha)^{s_1} (\lambda - \beta)^{s_2}$$

$$s_1 = \max\{n_1, n_2\}$$

$$s_2 = \max\{m_1, m_2\}$$

4. minimal poly

$$P_A(x) = (x-1)^3 (x-2)^2$$

of 5×5

so $\chi_A(x) = (x-1)^3 (x-2)^2$

$$\text{so} \quad J = \begin{bmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{bmatrix}_{5 \times 5}$$

5. $(x-1)^3 (x-2)^2 = \chi_A(x)$

then Jordan forms:

$P_A(x) = (x-1)(x-2)$

$$\begin{bmatrix} (1) & & & & \\ & (1) & & & \\ & & (1) & & \\ & & & (2) & \\ & & & & (2) \end{bmatrix}$$

$(x-1)(x-2)^2$

$$\begin{bmatrix} (1) & & & \\ & (1) & & \\ & & (1) & \\ & & & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{bmatrix}$$

$P_A(x) = (x-1)^2 (x-2)$

$$\begin{bmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & & & \\ & (1) & & \\ & & (2) & \\ & & & (2) \end{bmatrix}$$

$(x-1)^2 (x-2)^2$

$$\begin{bmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & & \\ & (1) & \\ & & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{bmatrix}$$

$p_A(x) = (x-1)^3 (x-2)$

$$\begin{bmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & & \\ & (2) & \\ & & (2) \end{bmatrix}$$

$(x-1)^3 (x-2)^2$

$$\begin{bmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & \\ & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{bmatrix}$$

Tutorial-10:

1. V is f.d IPS
for $x, y \in V$

To prove: $\langle x|y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|x+i^k y\|^2$

proof: we have
$\langle x|y\rangle = Re\langle x|y\rangle + i\, im\langle x|y\rangle$

now, $\|x+y\|^2 = \langle x+y|x+y\rangle$
$= \|x\|^2 + 2Re\langle x|y\rangle + \|y\|^2$
$\|x-y\|^2 = \|x\|^2 - 2Re\langle x|y\rangle + \|y\|^2$

$\Rightarrow \frac{\|x+y\|^2 - \|x-y\|^2}{4} = Re\langle x|y\rangle$

$\Rightarrow \frac{1}{4}\left[\|x+y\|^2 + i^2\|x+i^2y\|^2\right] = Re\langle x|y\rangle$

$\langle x|y\rangle = Re\langle x|y\rangle + i\, im\langle x|y\rangle$
$i\langle x|y\rangle = iRe\langle x|y\rangle - im\langle x|y\rangle$
$\langle x|iy\rangle = -iRe\langle x|y\rangle + im\langle x|y\rangle$
$= Re\langle x|iy\rangle + i\, im\langle x|iy\rangle$

$\Rightarrow Re\langle x|iy\rangle = im\langle x|y\rangle$

$\Rightarrow iRe\langle x|iy\rangle = \frac{i\|x+iy\|^2 + i^3\|x+i^3y\|^2}{4}$

now $\langle x|y\rangle = Re\langle x|y\rangle + i\, im\langle x|y\rangle$
$= Re\langle x|y\rangle + i[Re\langle x|iy\rangle]$
$\langle x|y\rangle = \frac{1}{4}\sum_{c=0}^{3} i^c\|x+i^c y\|^2$

2. V is IPS with ONB $\{x_1, \ldots\ldots x_n\} = B$
$\alpha_1, \alpha_2 \ldots \alpha_n \in F$ (scalars)

To find: $x \in V$ s.t $\langle x, x_i\rangle = \alpha_i \ \forall i = 1, \ldots n$

let $x = \sum_{i=1}^{n} c_i x_i$ then
$\langle x|x_i\rangle = c_i\langle x_i|x_i\rangle$
$= c_i$
putting $c_i = \alpha_i$ we get
$x = \sum \alpha_i x_i$ s.t
$\langle x|x_i\rangle = \alpha_i$

this is unique as if $x = \sum c_i x_i$
$x = \sum \alpha_i x_i$
then
$c_i = \alpha_i \ \forall i$

3. (a) $\langle A|B\rangle = tr(AB^*)$ is inner product on $M_{n\times n}(\mathbb{C})$:

① $\langle A+B|C\rangle = tr((A+B)C^*)$
$= tr(AC^* + BC^*)$
$= tr(AC^*) + tr(BC^*)$
$= \langle A|C\rangle + \langle B|C\rangle$

② $\langle A|A\rangle = tr(AA^*) = \sum_i \left(\sum_j a_{ij}\, a^*_{ji}\right)$
$= \sum_i \left(\sum_j a_{ij}\, \overline{a_{ij}}\right)$

as $a_{ij}\overline{a_{ij}} \geq 0 \Rightarrow \langle A|B\rangle \geq 0$

③ $\langle cA|B\rangle = tr(cAB^*)$
$= c\, tr(AB^*)$
$= c\langle A|B\rangle$

④ $\overline{\langle A|B\rangle} = \overline{tr(AB^*)} = tr(\overline{AB^*}) = tr(\overline{AB^*})^T$
$= tr((AB^*)^*)$
$= tr(BA^*)$
$= \langle B|A\rangle$

(b) $W = \text{span}\left\{ \begin{bmatrix} 1_0 & 0 \\ 0 & \ddots_0 \end{bmatrix}, \begin{bmatrix} 0_1 & 0 \\ & 1_0 & \\ 0 & & \ddots_0 \end{bmatrix}, \cdots \begin{bmatrix} 0 & \cdot & 0 \\ 0 & & \ddots 0_1 \end{bmatrix} \right\}$

$W^\perp = \{\alpha \in V \mid \langle \alpha|\beta\rangle = 0 \ \forall \beta \in W\}$
$\downarrow$
ortocomplimet of W

$\langle \alpha|\beta\rangle = 0 \ \forall B \in W$

$\Rightarrow \langle A|B\rangle = tr(AB^*) = 0 \ \forall B \in W$
as this is tree for every $B \in W$
for $B = \begin{bmatrix} 1_0 & 0 \\ 0 & \ddots_0 \end{bmatrix}$

$B^* = \begin{bmatrix} 1_0 & 0 \\ 0 & \ddots_0 \end{bmatrix}$

$AB^* = \begin{bmatrix} a_{11} & a_{12} & \cdots & \vdots \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ \vdots & \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} 1_0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$

$tr(AB^*) = a_{11} = 0$ or $a_{11} = 0$

Similary $a_{ii} = 0 \ \forall i$

$\therefore W^\perp =$ Subspace of all matrices with diagonal 0.

4. In $\mathbb{C}^3$

To find: orthonormal Basis (ONB) for

$$W = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1+i \end{pmatrix} \right\}$$

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} 2 \\ 1 \\ 1+i \end{pmatrix} - \frac{\langle (2,1,1+i) | (1,0,i) \rangle}{\langle (1,0,i) | (1,0,i) \rangle} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1+i \end{pmatrix} - \frac{\langle 2.1 + 1.0 + (1+i)(-i) \rangle}{\langle (1.1 + i.(-i) \rangle} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1+i \end{pmatrix} - \frac{(3-i)}{2} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} 2 - (3-i)/2 \\ 1 \\ 1+i - (3-i)i/2 \end{pmatrix} = \begin{pmatrix} 2 - 3/2 + i/2 \\ 1 \\ 1+i - 3i/2 - 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 + i/2 \\ 1 \\ 1/2 - i/2 \end{pmatrix}$$

$$\frac{\alpha_1}{\|\alpha_1\|} = \frac{(1,0,i)}{\sqrt{1.1 + i(-i)}} = \frac{(1,0,i)}{\sqrt{2}}$$

$$\frac{\alpha_2}{\|\alpha_2\|} = \frac{(1+i, 2, 1-i)}{\sqrt{1+1+4+1+1}} = \frac{(1+i, 2, 1-i)}{\sqrt{8}} = \frac{(1+i, 2, 1-i)}{2\sqrt{2}}$$

$$= \frac{(1/2 + i/2, 1, 1/2 - i/2)}{\sqrt{2}}$$

$$\therefore \text{ONB} = \left\{ \frac{\alpha_1}{\|\alpha_1\|}, \frac{\alpha_2}{\|\alpha_2\|} \right\}$$

(b) Best approximation of $(0,i,0)$ in W is :

$$\beta = \left\langle \frac{\alpha_1}{\|\alpha_1\|} \middle| (0,i,0) \right\rangle \frac{\alpha_1}{\|\alpha_1\|} + \left\langle \frac{\alpha_2}{\|\alpha_2\|} \middle| (0,i,0) \right\rangle \frac{\alpha_2}{\|\alpha_2\|}$$

$$= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \left\langle \left(\frac{1}{2}+\frac{i}{2}, 1, \frac{1}{2}-\frac{i}{2}\right) \middle| (0, i, 0) \right\rangle \left(\frac{1}{2}+\frac{i}{2}, 1, \frac{1}{2}-\frac{i}{2}\right)$$

$$= \frac{1}{2}(-i)\left(\frac{1+i}{2}, 1, \frac{1-i}{2}\right)$$

5. $M_{n\times n}(\mathbb{C})$ IPS

$$\langle A | B \rangle = \mathrm{tr}(AB^*) \quad \forall A, B \in M_{n\times n}(\mathbb{C})$$

$$T_P : M_{n\times n}(\mathbb{C}) \longrightarrow M_{n\times n}(\mathbb{C})$$
$$A \longmapsto P^{-1}AP$$

P is invt

now $\langle T_P(A) | B \rangle = \langle A | (T_P)^*(B) \rangle$

$$\mathrm{tr}(P^{-1}APB^*) = \langle T_P(A) | B \rangle$$

now

$$= \mathrm{tr}(APB^*P^{-1})$$
$$= \mathrm{tr}(A\,[(P^*)^{-1}BP^*]^*)$$
$$= \langle A | (P^*)^{-1}BP^* \rangle$$
$$= \langle A | T_{P^*}(B) \rangle = \langle A | (T_P)^*(B) \rangle$$

$$\Rightarrow T_{P^*} = (T_P)^*$$

6. (i) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

then $A^* = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = A$ $\therefore$ Normal

now $\det(A - xI) = (x-1)^2 - 1 = (x-1-1)(x-1+1)$
$= (x-2)(x)$

or $\exists$ Basis s.t

$$A \sim \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

now, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

then $\mathrm{Null}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \left\{ x \,\middle|\, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

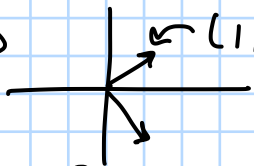
$x_1 + x_2 = 0$

or

$$\mathrm{Null}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

so one of the U row is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}\frac{1}{\sqrt{2}}$

now $(1,1)$ is $\perp$ to $(1,-1)$



so $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = U$ s.t $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \cdot & \cdot \\ - & \cdot \end{bmatrix}$

$$= \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

(ii) $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $B^* = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$\therefore$ B is normal

now, $(x-1)(x-1) - (-2)(-2) = 0$
$= (x-1)^2 - (2)^2$
$= (x-1-2)(x-1+2)$
$= (x-3)(x+1)$

$\chi_B(x) = (x+1)(x-3)$

Now $\text{Null}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

$\text{Null}\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

or $U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

7. Normal or $AA^* = A^*A$
and $A^m = 0$
for some $m \in \mathbb{N}$

then $m = 1$ (trivial)

for $m > 1$ we have

$AA^* = A^*A$
& $A^m = 0$
now as $AA^* = A^*A$
$\exists U$ s.t

$U^*AU = \Lambda$

diagonal

ortonormal Basis
(Unitary operator)

$U^*AU = \Lambda$
now
as A is nilpotent

$(U^*AU)^m = U^*A^mU = 0$
or
U^*AU is nilpotent
$\Rightarrow \Lambda^m = 0$
diagonal matrix $\Lambda^m = 0 \Rightarrow a_{ii}^m = 0$
$\Rightarrow a_{ii} = 0$
$\therefore$ Λ is 0 $\Rightarrow U^*AU = 0 \Rightarrow U^*A = 0 \Rightarrow A = 0$ $\therefore$ A is 0.